

LIPSCHITZ EMBEDDINGS OF METRIC SPACES INTO c_0

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Abstract. let M be a separable metric space. We say that $f = (f_n) : M \rightarrow c_0$ is a good- λ -embedding if, whenever $x, y \in M$, $x \neq y$ implies $d(x, y) \leq \|f(x) - f(y)\|$ and, for each n , $Lip(f_n) < \lambda$, where $Lip(f_n)$ denotes the Lipschitz constant of f_n . We prove that there exists a good- λ -embedding from M into c_0 if and only if M satisfies an internal property called $\pi(\lambda)$. As a consequence, we obtain that for any separable metric space M , there exists a good-2-embedding from M into c_0 . These statements slightly extend former results obtained by N. Kalton and G. Lancien, with simplified proofs.

1) Introduction.

First, let us recall that if f is a mapping between the metric spaces (M, d) and (N, δ) , the Lipschitz constant $Lip(f)$ is the infimum of all λ such that for all $(x, y) \in M^2$, $\delta(f(x), f(y)) \leq \lambda d(x, y)$.

Let (M, d) be a separable metric space and $\lambda \geq 1$. We say that $f : M \rightarrow c_0$ is a λ -embedding if, whenever $x, y \in M$, then :

$$(1) \quad d(x, y) \leq \|f(x) - f(y)\| \leq \lambda d(x, y).$$

Let us denote $f = (f_n)$ and, for each n , $E_n = \{(x, y) \in M \times M; d(x, y) \leq |f_n(x) - f_n(y)|\}$. Whenever $x, y \in M$, we have $\|f(x) - f(y)\| = \max_n |f_n(x) - f_n(y)|$. Hence f is a λ -embedding if for each n , $Lip(f_n) \leq \lambda$ and $M \times M = \bigcup_n E_n$.

I. Aharoni [1] proved that for any separable metric space M , there exists a λ -embedding from M into c_0 for any $\lambda > 6$, and that there is no λ -embedding from ℓ^1 into c_0 if $\lambda < 2$. P. Assouad [2] improved this result by showing that one can construct a λ -embedding from any separable metric space M into c_0 for any $\lambda > 3$. Later on, J. Pelant [4] obtained the same result with $\lambda = 3$. It was also observed that there is no λ -embedding from ℓ^1 into c_0^+ if $\lambda < 3$. All these authors actually constructed λ -embeddings into the positive cone c_0^+ of c_0 . Finally N. Kalton and G. Lancien [3] proved that for any separable metric space M , there exists a 2-embedding from M into c_0 , and this result is optimal (consider $M = \ell^1$).

We say that $f : M \rightarrow c_0$ is a strict- λ -embedding if, whenever $x, y \in M$ and $x \neq y$, then :

$$(2) \quad d(x, y) < \|f(x) - f(y)\| < \lambda d(x, y).$$

We say that $f = (f_n) : M \rightarrow c_0$ is a good- λ -embedding if, for each n , $Lip(f_n) < \lambda$, and $M \times M = \bigcup_n E_n$.

Proposition 1.1. *Assume $f : M \rightarrow c_0$ is a good- λ -embedding. Then, there exists $g : M \rightarrow c_0$ which is a strict and good- λ -embedding.*

Proof. Let $\lambda_n < \lambda$ be such that $f_n : M \rightarrow \mathbb{R}$ is λ_n -Lipschitz continuous, and let us define $g = (g_n) : M \rightarrow c_0$ such that for each n , $g_n = \alpha_n f_n$ with $1 < \alpha_n < 2$ and $\alpha_n \lambda_n < \lambda$. Clearly, g is still a good- λ -embedding. If $x \neq y$, since the sequences $(f_n(x))$ and $(f_n(y))$ tend to zero, the sequence $(g_n(x) - g_n(y))$ also converges to 0 and there exists n_0 such that

$$\|g(x) - g(y)\| = |g_{n_0}(x) - g_{n_0}(y)| \leq \alpha_{n_0} \lambda_{n_0} d(x, y) < \lambda d(x, y)$$

Since $\|f(x) - f(y)\| \leq \|g(x) - g(y)\|$, this implies that $\|f(x) - f(y)\| < \lambda d(x, y)$. On the other hand, let m_0 be such that $\|f(x) - f(y)\| = |f_{m_0}(x) - f_{m_0}(y)|$. We have

$$d(x, y) \leq \|f(x) - f(y)\| = |f_{m_0}(x) - f_{m_0}(y)| < |\alpha_{m_0} f_{m_0}(x) - \alpha_{m_0} f_{m_0}(y)| \leq \|g(x) - g(y)\|.$$

Therefore, g is also a strict- λ -embedding.

Our purpose is to prove that for every separable metric space, one can construct a strict-2-embedding from M into c_0 . We introduce also a property $\pi(\lambda)$ of a metric space, slightly weaker than a property introduced by N. Kalton and G. Lancien, and we prove that if $1 < \lambda \leq 2$, a separable metric space M admits a good- λ -embedding into c_0 if and only if it has the property $\pi(\lambda)$.

2) Necessary condition for the existence of good- λ -embedding into c_0 .

Let (M, d) be a metric space and E be a non empty subset of $M \times M$. We denote $\pi_1(E) = \{x \in M; \exists y \in M, (x, y) \in E\}$, $\pi_2(E) = \{y \in M; \exists x \in M, (x, y) \in E\}$ the projections of E , and $\pi(E) = \pi_1(E) \times \pi_2(E)$ the smallest rectangle containing E . We also define the gap of E by $\delta(E) := \inf\{d(x, y); (x, y) \in E\}$ and the diameter of E by $diam(E) = \sup\{d(x, y); (x, y) \in E\}$. These notions are not quite standard, and require some comments. Let us denote $\Delta := \{(x, x); x \in M\}$ the diagonal of $M \times M$, and let us endow the set $M \times M$ with the metric $d_1((x, y), (x', y')) = d(x, x') + d(y, y')$. The distance from a point $(y, z) \in M \times M$ to Δ is

$$d_1((y, z), \Delta) = \inf\{d_1((y, z), (x, x)); (x, x) \in \Delta\}$$

and it is easy to check that $d_1((y, z), \Delta) = d(y, z)$. Consequently, if $\emptyset \neq E \subset M \times M$, the smallest distance from a point of E to Δ is the quantity

$$d_1(E, \Delta) = \inf\{d_1((y, z), \Delta); (y, z) \in E\} = \delta(E)$$

On the other hand, the largest distance from a point of E to Δ is

$$D_1(E, \Delta) = \sup\{d_1((y, z), \Delta); (y, z) \in E\} = diam(E)$$

Whenever E is of the form $U \times V$, then $\delta(E) = \inf\{d(x, y); x \in U, y \in V\}$ is the gap between U and V , and $diam(E) = \sup\{d(x, y); x \in U, y \in V\}$. Thus, if $U = V$, $diam(E)$ is the usual diameter of U .

Fact 2.1. Let E be a bounded subset of $M \times M$, F be a finite dimensional normed vector space, let $P : M \rightarrow F$ be such that $\text{Lip}(P) \leq \lambda$ and $d(x, y) \leq \|P(x) - P(y)\|$ for each $(x, y) \in E$, and let $\varepsilon > 0$. Then, there exists a finite partition $\{E_1, \dots, E_N\}$ of E so that

$$\text{for each } n, \quad \text{diam}(E_n) < \lambda\delta(\pi(E_n)) + \varepsilon.$$

Proof. The set $P(\pi_1(E) \cup \pi_2(E)) \subset F$ is bounded as E is bounded and P , π_1 and π_2 are Lipschitz. Hence we can find a finite partition of this set into subsets F_j of diameter $< \varepsilon/4$. The sets $E_{j,k} = (P^{-1}(F_j) \times P^{-1}(F_k)) \cap E$ which are non empty form a partition of E . If $(x, y) \in E_{j,k}$ and $(u, v) \in \pi(E_{j,k})$, then

$$\|P(x) - P(y)\| \leq \|P(x) - P(u)\| + \|P(u) - P(v)\| + \|P(v) - P(y)\| \leq \varepsilon/2 + \lambda d(u, v)$$

Thus $d(x, y) \leq \lambda d(u, v) + \varepsilon/2$. The result follows by taking the infimum over all $(u, v) \in \pi(E_{j,k})$, the supremum over all $(x, y) \in E_{j,k}$, and by relabeling the sets $E_{j,k}$.

Definition 2.2. A metric space (M, d) has property $\pi(\lambda)$ if, for any balls B_1 and B_2 of radii r_1 and r_2 and for any non empty subset E of $B_1 \times B_2$ satisfying $\delta(E) > \lambda(r_1 + r_2)$, there exists a partition $\{E_1, \dots, E_N\}$ of E , such that

$$\text{for each } n, \quad \text{diam}(E_n) < \lambda\delta(\pi(E_n))$$

We say that (M, d) has the property weak- $\pi(\lambda)$ if the conclusion is replaced by the weaker conclusion : there exists non empty closed subsets F_1, \dots, F_N covering E such that

$$\text{for each } n, \quad r_1 + r_2 < \delta(\pi(F_n))$$

this conclusion is indeed weaker : if F_n is the closure of E_n in $M \times M$, then $\lambda(r_1 + r_2) < \delta(E) \leq \text{diam}(E_n) < \lambda\delta(\pi(E_n)) = \lambda\delta(\pi(F_n))$. It is also easy to see that if $\lambda < \mu$ and if M has $\pi(\lambda)$, then M has $\pi(\mu)$, and that if M has at least 2 elements, M never has $\pi(1)$.

Proposition 2.3. 1) Assume that there is a good- λ -embedding from M into c_0 . Then M has property $\pi(\lambda)$.

2) If (M, d) λ -embeds into c_0 , then M has property weak- $\pi(\lambda)$.

Proof. Let $f : M \rightarrow c_0$ be a λ -embedding. If (e_i) is the unit vector basis of c_0 , then $f(x) = \sum_{i=0}^{+\infty} f_i(x)e_i$. Let B_1 and B_2 be balls of radii r_1 and r_2 and of centers a_1 and a_2 , and $E \subset B_1 \times B_2$ such that $\delta(E) > \lambda(r_1 + r_2)$. We claim that the function $E \ni (x, y) \mapsto \|f(x) - f(y)\|$ depends on finitely many coordinates, i. e. there exists $i_0 \in \mathbb{N}$ such that, if $P(x) = \sum_{i=0}^{i_0} f_i(x)e_i$ then $\|f(x) - f(y)\| = \|P(x) - P(y)\|$.

Fix $\varepsilon > 0$ such that $\varepsilon < \delta(E) - \lambda(r_1 + r_2)$. We choose i_0 such that, if $Q = f - P$, then $\|Q(a_1) - Q(a_2)\| < \varepsilon$. If $(x, y) \in E$, then

$$\begin{aligned} \|Q(x) - Q(y)\| &\leq \|Q(x) - Q(a_1)\| + \|Q(a_1) - Q(a_2)\| + \|Q(a_2) - Q(y)\| \\ &< \lambda(r_1 + r_2) + \varepsilon < \delta(E) \leq d(x, y). \end{aligned}$$

Hence $d(x, y) \leq \|f(x) - f(y)\| = \max\{\|Q(x) - Q(y)\|, \|P(x) - P(y)\|\} = \|P(x) - P(y)\|$. This proves our claim. Since $\text{Lip}(P) \leq \lambda$, Fact 2.1 implies the existence of a partition $\{E_1, \dots, E_N\}$ of E such that for all n , $\text{diam}(E_n) < \lambda\delta(\pi(E_n)) + \varepsilon$. Since we also have $\lambda(r_1 + r_2) + \varepsilon \leq \text{diam}(E_n)$, we have $r_1 + r_2 < \delta(\pi(E_n))$, so if F_n is the norm closure of E_n in E ,

$$r_1 + r_2 < \delta(\pi(F_n))$$

When f is a good- λ -embedding, the mapping P is μ -Lipschitz continuous for some $\mu < \lambda$, so we can assume that for all n , $\text{diam}(E_n) < \mu\delta(\pi(E_n)) + \alpha$, where $\alpha = \min\{\varepsilon, (\lambda - \mu)(r_1 + r_2)\}$. This still implies $r_1 + r_2 < \delta(\pi(E_n))$. Finally,

$$\text{diam}(E_n) < \mu\delta(\pi(E_n)) + (\lambda - \mu)(r_1 + r_2) < \lambda\delta(\pi(E_n)).$$

Corollary 2.4 (see [3]). *Let X be a Banach space. If there exists $u \in S_X$ and an infinite dimensional subspace Y of X such that $\inf\{\|u + y\|; y \in S_Y\} > \lambda$, then there is no λ -embedding from X into c_0 .*

Proof. If $E = \{(u + y, -u - y); y \in S_Y\} \subset \overline{B}(u, 1) \times \overline{B}(-u, 1)$, E satisfies $\delta(E) > 2\lambda$. Assume there exists a λ -embedding from M into c_0 . Then X has the weak- $\pi(\lambda)$ property, so there exists closed subsets F_1, \dots, F_N of E covering E such that for each n , $\delta(\pi(F_n)) > 2$. On the other hand, $A_n = \{y \in S_Y; (u + y, -u - y) \in F_n\}$ is closed and $A_1 \cup \dots \cup A_N = S_Y$. Since $\dim(Y) > N$, the Borsuk-Ulam theorem yields the existence of $y \in S_Y$ and n such that $\{y, -y\} \subset A_n$. Hence $(u + y, -u - y) \in \pi(F_n)$ and so $\delta(\pi(F_n)) \leq 2$, which is absurd.

Example 2.5. *There is no λ -embedding from ℓ^p into c_0 for any $\lambda < 2^{1/p}$. In particular, ℓ^1 is a metric space which does not λ -embed into c_0 with $\lambda < 2$. (If $u = e_0$ and $Y = \{y = (y_i) \in \ell^p; y_0 = 0\}$, then $\|u + y\| = 2^{1/p}$ for all $y \in S_Y$).*

3) Examples of spaces with property $\pi(\lambda)$.

Example 3.1. *A metric space such that the bounded subsets of M are totally bounded has property $\pi(1 + \varepsilon)$ for all $\varepsilon > 0$ (partition E into subsets E_n of small d_1 -diameter).*

Example 3.2. *If (M, d) is a metric space, then (M, d) has property $\pi(2)$.*

Therefore, property $\pi(\lambda)$ is of interest only if $1 < \lambda \leq 2$.

Proof. Let $E \subset B_1 \times B_2$, with B_1 et B_2 balls of radii $r_1 \geq r_2$, and assume $\varepsilon := \delta(E) - 2(r_1 + r_2) > 0$. Let $a_0 = \delta(E) < a_1 < \dots < a_{N-1} < \text{diam}(E) < a_N$ so that for all $1 \leq n \leq N$, $a_n - a_{n-1} < \varepsilon$. Define, for $1 \leq n \leq N$, $E_n = \{(x, y) \in E; a_{n-1} \leq d(x, y) < a_n\}$. Thus $\delta(E_n) + \varepsilon > \text{diam}(E_n)$. If $(u, v) \in \pi(E_n)$, one can find $v' \in B_2$ such that $(u, v') \in E_n$. Moreover $v, v' \in B_2$, hence :

$$\text{diam}(E_n) < 2\delta(E_n) - 2(r_1 + r_2) \leq 2d(u, v') - 2d(v', v) \leq 2d(u, v)$$

Taking the infimum over all $(u, v) \in \pi(E_n)$, we get $\text{diam}(E_n) < 2\delta(\pi(E_n))$.

Example 3.3. If (X_n) is a sequence of finite dimensional Banach spaces, then $(\oplus X_n)_p$ has property $\pi(2^{1/p})$.

Proof. Let $E \subset B(a_1, r_1) \times B(a_2, r_2)$ such that $\alpha = \delta(E)^p - 2(r_1 + r_2)^p > 0$. We select $\varepsilon > 0$ so that $(r_1 + r_2 + \varepsilon)^p < \delta(E)^p/2 - \alpha/4$ and $2(t + \varepsilon)^p - \alpha/2 < 2t^p$ whenever $0 \leq t \leq \text{diam}(E)$. If $x \in (\oplus X_n)_p$, then $x = \sum_{n=1}^{\infty} x_n$ with $x_n \in X_n$ for each n . Define

$P, Q : (\oplus X_n)_p \rightarrow (\oplus X_n)_p$ by $P(\sum_{i=0}^{\infty} x_i) = \sum_{i=0}^{i_0} x_i$ and $Q = I - P$, where i_0 is such that $\|Qa_1 - Qa_2\| < \varepsilon$. According to Fact 2.1, since P is an operator of norm 1 with values in a finite dimensional subspace of $(\oplus X_n)_p$, we can find relatively closed subsets E_n of E covering E such that, for all n , if $(x, y) \in E_n$, then $\|Px - Py\| \leq \delta(\pi(E_n)) + \varepsilon$. On the other hand, $\|Qx - Qy\| \leq \|Qx - Qa_1\| + \|Qy - Qa_2\| + \|Qa_1 - Qa_2\| \leq r_1 + r_2 + \varepsilon$. Moreover, $\|x - y\|^p = \|Px - Py\|^p + \|Qx - Qy\|^p$, so

$$\text{diam}(E_n)^p \leq (\delta(\pi(E_n)) + \varepsilon)^p + (r_1 + r_2 + \varepsilon)^p < (\delta(\pi(E_n)) + \varepsilon)^p + \text{diam}(E_n)^p/2 - \alpha/4$$

which implies $\text{diam}(E_n)^p < 2(\delta(\pi(E_n)) + \varepsilon)^p - \alpha/2 < 2\delta(\pi(E_n))^p$.

Remark 3.4. In the definition of property $\pi(\lambda)$, if a_1 and a_2 are centers of B_1 and B_2 , we can assume that

$$(\lambda - 1)(r_1 + r_2) < d(a_1, a_2) \leq \frac{\lambda + 1}{\lambda - 1}(r_1 + r_2)$$

Indeed, if $E \neq \emptyset$, then, for each $(x, y) \in E$, $\lambda(r_1 + r_2) < \delta(E) \leq d(x, y) \leq d(a_1, a_2) + (r_1 + r_2)$, which proves the first inequality. If $\frac{\lambda+1}{\lambda-1}(r_1 + r_2) < d(a_1, a_2)$, the conclusion of property $\pi(\lambda)$ is always true if we take $N = 1$ and $E_1 = E$, since then

$$\text{diam}(E) \leq d(a_1, a_2) + (r_1 + r_2) < \lambda(d(a_1, a_2) - (r_1 + r_2)) \leq \lambda\delta(\pi(E))$$

4) Constructing strict- λ -embeddings into c_0 .

Theorem 4.1. Let (M, d) be a separable metric space and $1 < \lambda \leq 2$. If M has property $\pi(\lambda)$, there exists a λ -embedding from M into c_0 which is strict and good.

Theorem 4.1 and Proposition 2.3 show that there exists a good- λ -embedding from M into c_0 if and only if M has property $\pi(\lambda)$. We do not know any internal characterization of separable metric spaces that admit a λ -embedding into c_0 .

Corollary 4.2. Let (M, d) be a metric space such that the bounded subsets of M are totally bounded. For all $\varepsilon > 0$, there exists a $(1 + \varepsilon)$ -embedding from M into c_0 .

Corollary 4.3. Every separable metric space strictly-2-embeds into c_0 .

This result is optimal since ℓ^1 does not λ -embed into c_0 whenever $\lambda < 2$.

Corollary 4.4. *If (X_n) is a sequence of finite dimensional Banach spaces, then $(\oplus X_n)_p$ strictly- $2^{1/p}$ -embeds into c_0 .*

This result is optimal since we have seen that there is no λ -embedding from ℓ^p into c_0 with $\lambda < 2^{1/p}$. We now turn to the proof of Theorem 4;1. We need some further notations. If (M, d) is a metric space, $x \in M$ and $U, V \subset M$, the distance from x to U is $d(x, U) = \delta(\{x\} \times U)$ and the gap between U and V is $\delta(U, V) = \delta(U \times V)$. The coordinates of the embedding from M into c_0 are of the following type :

Lemma 4.5. *Let (M, d) be a metric space, U, V, F three non empty subsets of M and $\varepsilon \geq 0$. There exists $f : M \rightarrow \mathbb{R}$, 1-Lipschitz, such that :*

- 1) For all $x \in F$, $|f(x)| \leq \varepsilon$.
- 2) For all $(x, y) \in U \times V$, $f(x) - f(y) = \min \{ \delta(U, V), \delta(U, F) + \delta(V, F) + 2\varepsilon \}$.

Lemma 4.6. *Let $1 < \lambda \leq 2$, (M, d) be a metric space with property $\pi(\lambda)$, $F \subset G$ be finite subsets of M and $0 < \alpha < \beta$. We set :*

$$A(F, \beta) = \{ (x, y) \in M \times M; \lambda(d(x, F) + d(y, F) + \beta) \leq d(x, y) \}$$

Then there exists a finite partition $\{E_1, \dots, E_N\}$ of $A(G, \alpha) \setminus A(F, \beta)$ such that, if we denote $\pi(E_n) = U_n \times V_n$ then

$$\text{for each } n \quad \text{diam}(E_n) < \lambda \min \{ \delta(U_n, V_n), \delta(U_n, F) + \delta(V_n, F) + 2\beta \}.$$

Proof of Theorem 4.1.

The goal is to construct a sequence (f_n) of 1-Lipschitz continuous functions satisfying, for every $x \in M$, $\lim_{n \rightarrow \infty} f_n(x) = 0$, and a partition $\{E_n; n \in \mathbb{N}\}$ of $\{(x, y) \in M \times M; x \neq y\}$, so that for each n , the function $(x, y) \rightarrow f_n(x) - f_n(y)$ is equal to some constant c_n on E_n and $\text{diam}(E_n) < \lambda c_n$. The required strict and good- λ -embedding is then $f = (\lambda_n f_n)$ where $\lambda_n < \lambda$ is chosen so that $\text{diam}(E_n) < \lambda_n c_n$.

Let (a_k) be a dense sequence of distinct points of M , $F_k = \{a_1, \dots, a_k\}$, and (ε_k) be a decreasing sequence of real numbers converging to 0. We set $\Delta_k = A(F_{k+1}, \varepsilon_{k+1}) \setminus A(F_k, \varepsilon_k)$. The sets Δ_k form a partition of $\{(x, y) \in M \times M; x \neq y\}$. Indeed, if $x, y \in M$, $x \neq y$ and if $\sigma_k = \lambda(d(x, F_k) + d(y, F_k) + \varepsilon_k)$, then $0 < d(x, y) < \sigma_1$, (σ_k) is decreasing and $\lim_{k \rightarrow \infty} \sigma_k = 0$, so there exists a unique k such that $\sigma_{k+1} \leq d(x, y) < \sigma_k$, which means $(x, y) \in \Delta_k$.

By Lemma 4.6, there exists integers $0 = n_1 < n_2 < \dots < n_k < \dots$ and subsets E_n of $M \times M$ such that for all k , $\{E_n; n_k < n \leq n_{k+1}\}$ is a partition of Δ_k , and, whenever $n_k < n \leq n_{k+1}$ then

$$\text{diam}(E_n) < \lambda \min \{ \delta(U_n, V_n), \delta(U_n, F_k) + \delta(V_n, F_k) + 2\varepsilon_k \}.$$

where $\pi(E_n) = U_n \times V_n$. In particular, $\{E_n; n \in \mathbb{N}\}$ is a partition of $\{(x, y) \in M \times M; x \neq y\}$. By Lemma 4.5, there are 1-Lipschitz functions $f_n : M \rightarrow \mathbb{R}$ so that

- 1) if $x \in F_k$ and $n_k < n \leq n_{k+1}$, then $|f_n(x)| \leq \varepsilon_k$,
- 2) if $n_k < n \leq n_{k+1}$ and $(x, y) \in U_n \times V_n$, then

$$f_n(x) - f_n(y) = c_n := \min \{ \delta(U_n, V_n), \delta(U_n, F_k) + \delta(V_n, F_k) + 2\varepsilon_k \}.$$

and so $\text{diam}(E_n) < \lambda c_n$.

If $x \in M$, let us show that $\lim_{n \rightarrow \infty} f_n(x) = 0$. If $\varepsilon > 0$, we fix j such that $d(x, a_j) < \varepsilon/2$, then $k \geq j$ such that $\varepsilon_k < \varepsilon/2$. Since the functions f_n are 1-Lipschitz continuous, if $n \geq n_k$, then $|f_n(x)| \leq d(x, a_j) + |f_n(a_j)| < \varepsilon/2 + \varepsilon_k < \varepsilon$.

Proof of Lemma 4.5. We fix s, t such that $-\delta(V, F) - \varepsilon \leq s \leq 0 \leq t \leq \delta(U, F) + \varepsilon$ and $t - s = \min \{ \delta(U, V), \delta(U, F) + \delta(V, F) + 2\varepsilon \}$, and we set

$$f(x) := \min \{ d(x, U) + t, d(x, V) + s, d(x, F) + \varepsilon \}$$

The function f is 1-Lipschitz continuous as the infimum of 1-Lipschitz continuous functions.

If $x \in U$, $f(x) = \min \{ t, d(x, V) + s, d(x, F) + \varepsilon \} = t$ because $d(x, V) + s \geq \delta(U, V) + s \geq t$ and $d(x, F) + \varepsilon \geq \delta(U, F) + \varepsilon \geq t$. If $y \in V$, $f(y) = \min \{ d(y, U) + t, s, d(y, F) + \varepsilon \} = s$ because $s \leq 0$. Therefore, if $x \in U$ and $y \in V$, then $f(x) - f(y) = t - s$, which proves 2).

Finally, if $x \in F$, then $f(x) = \min \{ d(x, U) + t, d(x, V) + s, \varepsilon \} \leq \varepsilon$. On the other hand $d(x, U) + t \geq 0$ et $d(x, V) + s \geq \delta(V, F) + s \geq -\varepsilon$, so $f(x) \geq -\varepsilon$, which proves 1).

Proof of Lemma 4.6.

Set $\Delta := A(G, \alpha) \setminus A(F, \beta)$. There is a bounded subset B of M such that $\Delta \subset B \times B$, because $\lambda > 1$, G is bounded and

$$\lambda(d(x, G) + d(y, G)) \leq d(x, y) \leq d(x, G) + d(y, G) + \text{diam}(G).$$

whenever $(x, y) \in A(G, \alpha)$. Thus, there is a partition $\{B_1, B_2, \dots, B_m\}$ of the bounded set B such that for all j , if $x, x' \in B_j$ and $a \in G$, then $|d(x, a) - d(x', a)| \leq \alpha/5$, and so,

$$\text{for all } x \in B_j, \quad \text{for all } a \in G, \quad d(x, a) < d(B_j, a) + \alpha/4.$$

Since G is finite, there exists $a_j \in G$ such that $d(B_j, a_j) = \delta(B_j, G)$, and so $B_j \subset B(a_j, r_j)$, where $r_j = \delta(B_j, G) + \alpha/4$. The subsets $E_{jk} = \Delta \cap B_j \times B_k$ of Δ form a partition of Δ , $E_{jk} \subset B(a_j, r_j) \times B(a_k, r_k)$, and, if $(x, y) \in E_{jk}$:

$$\begin{aligned} d(x, y) &\geq \lambda(d(x, G) + d(y, G) + \alpha) \geq \lambda(\delta(B_j, G) + \delta(B_k, G) + \alpha) \\ &= \lambda(r_j + r_k + \alpha/2) > \lambda(r_j + r_k). \end{aligned}$$

So $\delta(E_{jk}) > \lambda(r_j + r_k)$. According to property $\pi(\lambda)$ applied to each E_{jk} , there exists a finite partition $\{E_1, \dots, E_N\}$ of Δ such that,

$$\text{diam}(E_n) < \lambda\delta(\pi(E_n)) = \lambda\delta(U_n, V_n),$$

where $\pi(E_n) = U_n \times V_n$. Moreover, if j, k, n are such that $E_n \subset B_j \times B_k$ and if $(x, y) \in E_n$, then

$$\begin{aligned} d(x, y) &\leq \lambda(d(x, F) + d(y, F) + \beta) \leq \lambda(\delta(B_j, F) + \alpha/4 + \delta(B_k, F) + \alpha/4 + \beta) \\ &\leq \lambda(\delta(U_n, F) + \delta(V_n, F) + \alpha/2 + \beta). \end{aligned}$$

hence

$$\text{diam}(E_n) \leq \lambda(\delta(U_n, F) + \delta(V_n, F) + \alpha/2 + \beta) < \lambda(\delta(U_n, F) + \delta(V_n, F) + 2\beta).$$

5) Some consequences.

Observe that a metric space has property $\pi(\lambda)$ (resp. weak- $\pi(\lambda)$) if and only if its bounded subsets have it. In particular a Banach space has property $\pi(\lambda)$ (resp. weak- $\pi(\lambda)$) if and only if its unit ball has it. Since the property “there exists a good- λ -embedding from M into c_0 ” is equivalent to the property “ M has $\pi(\lambda)$ ”, we can state :

Proposition 5.1. *Assume that (M, d) is a separable metric space and that for each ball B of M , there is a good- λ -embedding from B into c_0 . Then there is a strict and good- λ -embedding from M into c_0 .*

In particular, if X is a Banach space and if there exists a good- λ -embedding from its closed unit ball into c_0 , then there exists a good- λ -embedding from X into c_0 . The following extension result is obvious.

Proposition 5.2. *Assume that (M, d) is a separable metric space and that N is a dense subset of M . If there is a good- λ -embedding from N into c_0 , then there is a good- λ -embedding from M into c_0 .*

Remark 5.3. In Definition 2.2, we didn’t specify if the balls were closed or open. We can define two different properties, $\pi(\lambda)$ with closed balls and $\pi(\lambda)$ with open balls. These two properties are equivalent! Indeed, the proof of Proposition 2.3 shows that if there is a good- λ -embedding from M into c_0 , then M has property $\pi(\lambda)$ with closed balls, which in turn implies that M has property $\pi(\lambda)$ with open balls. On the other hand, the proof of Theorem 4.1 shows that if M has property $\pi(\lambda)$ with open balls, then there is a good- λ -embedding from M into c_0 . This proves that property $\pi(\lambda)$ with open balls is equivalent to property $\pi(\lambda)$ with closed balls.

N. Kalton and G. Lancien introduced the following definition : A metric space (M, d) has $\Pi(\lambda)$ if, for every $\mu > \lambda$, there exists $\nu > \mu$ such that for every closed balls B_1 and B_2

with positive radii r_1 and r_2 , there exists subsets $U_1, \dots, U_N, V_1, \dots, V_n$ of M such that the sets $U_n \times V_n$ are a covering of $E_\mu := \{(x, y) \in B_1 \times B_2; d(x, y) > \mu(r_1 + r_2)\}$ and,

$$\text{for all } n, \quad \lambda \delta(U_n, V_n) \geq \nu(r_1 + r_2)$$

Lemma 5.4. *Property $\Pi(\lambda)$ implies property $\pi(\lambda)$.*

We do not know if the converse is true. Let us notice that N. Kalton and G. Lancien proved that if a separable metric space satisfies property $\Pi(\lambda)$, then there exists $f : M \rightarrow c_0$ such that, for all $x, y \in M$, $x \neq y$, we have

$$d(x, y) < \|f(x) - f(y)\| \leq \lambda d(x, y)$$

which is weaker than the condition f is a strict and good- λ -embedding. Theorem 1 improves their result since our hypothesis, M has $\pi(\lambda)$, is weaker, and our conclusion, f is a strict and good- λ -embedding, is stronger. Moreover, our condition $\pi(\lambda)$ is a necessary and sufficient condition for the existence of a good- λ -embedding.

Proof of Lemma 5.4. Let us assume that (M, d) has property $\Pi(\lambda)$. Let $E \subset B_1 \times B_2$ such that $\delta(E) > \lambda(r_1 + r_2)$. We fix $\mu > \lambda$ such that $\delta(E) > \mu(r_1 + r_2)$. Then $E \subset E_\mu$. Let $\nu > \mu$ be given by property $\Pi(\lambda)$. Let $1 = a_1 < a_2 < \dots < a_K$ be a sequence such that $\text{diam}(E_\mu) = a_K \mu(r_1 + r_2)$ and $\frac{a_{k+1}}{a_k} < \frac{\nu}{\mu}$ whenever $1 \leq k < K$. We denote

$$E_k := \{(x, y) \in B_1 \times B_2; a_k \mu(r_1 + r_2) < d(x, y) \leq \mu(r_1 + r_2) a_{k+1}\}.$$

The E_k 's form a covering of E_μ . Let B_1^k and B_2^k be the closed balls of the same center as B_1 and B_2 and of radius $a_k r_1$ and $a_k r_2$ respectively. Obviously, $E_k \subset B_1^k \times B_2^k$. Applying property $\Pi(\lambda)$ for each k , we can find subsets $U_{k,1}, \dots, U_{k,N_k}, V_{k,1}, \dots, V_{k,N_k}$ of M such that the sets $U_{k,n} \times V_{k,n}$ for $1 \leq n \leq N_k$ form a covering of E_k and

$$\text{for all } n, \quad \lambda \delta(U_{k,n}, V_{k,n}) \geq \nu(a_k r_1 + a_k r_2) > \mu a_{k+1}(r_1 + r_2)$$

We can assume in addition that for each n , the sets $U_{k,n} \times V_{k,n}$ are pairwise disjoint (because a finite union of products can always be written as a finite union of pairwise disjoint products). If we denote $E_{k,n} = E \cap E_k \cap (U_{k,n} \times V_{k,n})$, the $E_{k,n}$'s form a partition of E . Moreover, $\pi(E_{k,n}) \subset U_{k,n} \times V_{k,n}$, and the above inequality implies

$$\text{for all } n, \quad \lambda \delta(\pi(E_{k,n})) > \mu a_{k+1}(r_1 + r_2) \geq \text{diam}(E_k) \geq \text{diam}(E_{k,n}).$$

We have proved property $\pi(\lambda)$.

6) Strict- λ -embeddings into c_0^+ .

Here, c_0^+ denotes the positive cone of c_0 . Observe that :

$$u, v \in c_0^+ \quad \Rightarrow \quad \|u - v\| \leq \max\{\|u\|, \|v\|\}.$$

The existence of a strict- λ -embedding into c_0^+ follows from the following property $\pi^+(\lambda)$.

Definition 6.1. A metric space (M, d) has property $\pi^+(\lambda)$ (with $\lambda > 1$) if,

a) Whenever B_1 and B_2 are balls of positive radii r_1 et r_2 and E is a subset of $B_1 \times B_2$ such that $\delta(E) > \lambda \max(r_1, r_2)$, there exists a finite partition $\{E_1, \dots, E_N\}$ of E satisfying

$$\text{for each } n, \quad \text{diam}(E_n) < \lambda \delta(\pi(E_n))$$

b) There exists $\theta < \lambda$ and $\varphi : M \rightarrow [0, +\infty[$ such that

$$|\varphi(x) - \varphi(y)| \leq d(x, y) \leq \theta \max(\varphi(x), \varphi(y)) \quad \text{for all } x, y \in M.$$

The function φ is called a control function.

Remark 6.2. 1) It is easy to see that it is enough to check a) whenever $r_1 = r_2 (= r)$.

2) If $\lambda > 2$, the function $\varphi(x) = d(x, a)$ is a control function (take $\theta = 2$). Therefore, the metric space M has property $\pi^+(\lambda)$ if and only if the bounded subsets of M have property $\pi^+(\lambda)$. In particular a Banach space X has property $\pi^+(\lambda)$ (with $\lambda > 2$) if and only if its unit ball has property $\pi^+(\lambda)$.

3) If M is bounded, then, for any $\lambda > 1$, the function $\varphi : M \rightarrow [0, +\infty[$ given by $\varphi(x) = d(x, a) + \text{diam}(M)$ satisfies condition b) of property $\pi^+(\lambda)$ (take $\theta = 1$).

Proposition 6.3. 1) If there is a λ -embedding from (M, d) into c_0^+ , then M has property $\pi^+(\mu)$ for all $\mu > \lambda$.

2) Assume that M is bounded or that $\lambda > 2$. If there is a good- λ -embedding from M into c_0 , then M has property $\pi(\lambda)$.

Proof. Let $B_1 = B(a_1, r)$ and $B_2 = B(a_2, r)$. Let $E \subset B_1 \times B_2$ such that $\lambda r + \varepsilon < \delta(E)$ for some $\varepsilon > 0$. Let $f : M \rightarrow c_0$ be a λ -embedding given by $f(x) = \sum_{i=0}^{+\infty} f_i(x) e_i$. We denote

$P(x) = \sum_{i=0}^{i_0} f_i(x) e_i$ and $Q = f - P$, where i_0 is such that $\max\{\|Q(a_1)\|, \|Q(a_2)\|\} < \varepsilon$. If $(x, y) \in E$, then

$$\begin{aligned} \|Q(x) - Q(y)\| &\leq \max\{\|Q(x)\|, \|Q(y)\|\} \\ &\leq \max\{\|Q(x) - Q(a_1)\| + \|Q(a_1)\|, \|Q(y) - Q(a_2)\| + \|Q(a_2)\|\} \\ &\leq \lambda r + \varepsilon < \delta(E) \leq d(x, y) \leq \|f(x) - f(y)\|. \end{aligned}$$

Thus, $\|f(x) - f(y)\| = \|P(x) - P(y)\|$. Following the lines of the proof of Proposition 2.3 we get, for any $\mu > \lambda$, a partition $\{E_1, \dots, E_N\}$ of E such that for each n , $\text{diam}(E_n) \leq \mu \delta(\pi(E_n))$. Condition a) can be checked and the E_n 's satisfy also $\delta(\pi(E_n)) > r$. Moreover, if $\varphi(x) = \|f(x)\|/\lambda$, then $|\varphi(x) - \varphi(y)| \leq d(x, y) \leq \lambda \max(\varphi(x), \varphi(y))$ for all $x, y \in M$. This proves that φ is a control function of $\pi^+(\mu)$ because $\lambda < \mu$.

The proof of 2) also follows the lines of the corresponding case of Proposition 2.3, and here we do not have to worry about the existence of a control function by Remark 6.2.

Corollary 6.4. *Let X be a Banach space. If there exists $u \in S_X$ and an infinite dimensional subspace Y of X such that $\inf\{\|u + 2y\|; y \in S_Y\} > \lambda$, then there is no λ -embedding from M into c_0^+ .*

Proof. If $E = \{(u + 2y, -u - 2y); y \in S_Y\} \subset \overline{B}(u, 2) \times \overline{B}(-u, 2)$, E satisfies $\delta(E) > 2\lambda$. If there is a λ -embedding from X into c_0^+ , then, by Proposition 6.3, there is a partition $\{E_1, \dots, E_N\}$ of E such that for each n , $\delta(\pi(E_n)) > 2$. If F_n is the norm closure of E_n , then $\{F_1, \dots, F_N\}$ is a covering of E and we still have $\delta(\pi(F_n)) > 2$.

But as in the proof of Corollary 2.4, we also have $\delta(\pi(F_n)) \leq 2$, which is absurd.

Example 6.5. *The metric space ℓ^1 does not λ -embed into c_0^+ whenever $\lambda < 3$. (If $u = e_0$ and $Y = \{y = (y_i) \in \ell^1; y_0 = 0\}$, then $\|u + 2y\| = 3$ for each $y \in S_Y$).*

The space ℓ^p does not λ -embed into c_0^+ whenever $\lambda < (1 + 2^p)^{1/p}$ (since in this case, for all $y \in S_Y$, $\|u + 2y\| = (1 + 2^p)^{1/p}$).

The balls of positive radius of ℓ^p do not λ -embed into c_0^+ whenever $\lambda < (1 + 2^p)^{1/p}$. On the other hand, it follows from Corollary 4.4 that the balls of positive radius of ℓ^p embed in c_0 if $\lambda = 2^{1/p} < (1 + 2^p)^{1/p}$.

Indeed, assume that for some $\lambda < (1 + 2^p)^{1/p}$, a ball B of positive radius of ℓ^p λ -embeds into c_0^+ . We can assume that B is the unit ball of ℓ^p . According to Proposition 6.3, B has property $\pi^+(\mu)$ for every $\mu > \lambda$, and by Remark 6.2 2), ℓ^p has property $\pi^+(\mu)$ for every $\mu > \lambda$, and from Theorem 6.9 below, ℓ^p μ -embeds into c_0^+ for every $\mu > \lambda$. But this contradicts the fact that ℓ^p does not μ -embed into c_0^+ whenever $\mu < (1 + 2^p)^{1/p}$.

Example 6.6. *A compact metric space M has property $\pi^+(\lambda)$ for all $\lambda > 1$. A metric space M such that its bounded subsets are totally bounded has property $\pi^+(\lambda)$ for all $\lambda > 2$, but may fail property $\pi^+(2)$.*

A metric space M such that its bounded subsets are totally bounded satisfies condition a) of property $\pi^+(\lambda)$ for all $\lambda > 1$, and any metric space satisfies condition b) of property $\pi^+(\lambda)$ for all $\lambda > 2$. Moreover, if $\lambda > 1$ and M is compact, then M is bounded and so satisfies condition b).

The bounded subsets of the set \mathbf{Z} of integers are finite. Let $\varphi : \mathbf{Z} \rightarrow [0, +\infty[$ such that $|\varphi(x) - \varphi(y)| \leq |x - y|$ for all $x, y \in \mathbf{Z}$. If $x_n = (-1)^n n$, then $|x_n - x_{n+1}| = 2n + 1$ and $\varphi(x_n) \leq \varphi(0) + n$. Consequently, if $\theta < 2$, then $|x_n - x_{n+1}| > \theta \max\{\varphi(x_n), \varphi(x_{n+1})\}$ for n large enough. Thus \mathbf{Z} do not admit any control function φ for property $\pi^+(2)$.

Example 6.7. *Each metric space M has property $\pi^+(3)$.*

Proof. Let B_1 and B_2 be balls of radius r and $E \subset B_1 \times B_2$ such that $\varepsilon := \delta(E) - 3r > 0$. As in Example 2, using the fact, there is a partition $\{E_1, \dots, E_N\}$ of E such that for each n , $\delta(E_n) + \varepsilon > \text{diam}(E_n)$. If $(u, v) \in \pi(E_n)$, there is $v' \in B_2$ so that $(u, v') \in E_n$. Moreover $v, v' \in B_2$, hence :

$$3d(u, v) \geq 3d(u, v') - 3d(v', v) \geq 3\delta(E_n) - 6r \geq \delta(E_n) + 2\varepsilon > \text{diam}(E_n)$$

Taking the infimum over all $(u, v) \in \pi(E_n)$, we get $3\delta(\pi(E_n)) > \text{diam}(E_n)$.

Example 6.8. $\ell^p(\mathbb{N})$ has property $\pi^+((1+2^p)^{1/p})$.

Proof. Let $E \subset B(a_1, r) \times B(a_2, r)$ such that $\alpha = \delta(E)^p - (1+2^p)r^p > 0$. We choose $\varepsilon > 0$ such that $(2r + \varepsilon)^p < \frac{2^p}{1+2^p}\delta(E)^p - \alpha/2$ and $(t + \varepsilon)^p - \alpha/2 < t^p$ si $0 \leq t \leq \text{diam}(E)$.

Let (e_i) be the canonical basis of ℓ^p , $P, Q : \ell^p \rightarrow \ell^p$ be defined by $P(\sum_{i=0}^{\infty} x_i e_i) = \sum_{i=0}^{i_0} x_i e_i$ and $Q = I - P$, where i_0 is chosen so that $\|Qa_1 - Qa_2\| < \varepsilon$. Since $\text{Lip}(P) = 1$ and P has its values in a finite dimensional space, Fact 2.1 implies the existence of a partition $\{E_1, \dots, E_N\}$ of E such that, for each n , if $(x, y) \in E_n$, then $\|Px - Py\| \leq \delta(\pi(E_n)) + \varepsilon$. On the other hand, $\|Qx - Qy\| \leq \|Qx - Qa_1\| + \|Qy - Qa_2\| + \|Qa_1 - Qa_2\| \leq 2r + \varepsilon$. Hence

$$\text{diam}(E_n)^p \leq (\delta(\pi(E_n)) + \varepsilon)^p + (2r + \varepsilon)^p \leq (\delta(\pi(E_n)) + \varepsilon)^p + \frac{2^p}{1+2^p}\delta(E)^p - \alpha/2$$

and so $\text{diam}(E_n)^p \leq (1+2^p)(\delta(\pi(E_n)) + \varepsilon)^p - (1+2^p)\alpha/2 < (1+2^p)\delta(\pi(E_n))^p$.

Theorem 6.9. *If the separable metric space (M, d) has property $\pi^+(\lambda)$ with $1 < \lambda \leq 3$, then there exists $f : M \rightarrow c_0^+$ such that for all $x, y \in M$, $x \neq y$, we have :*

$$d(x, y) < \|f(x) - f(y)\| < \lambda d(x, y).$$

Corollary 6.10. *Let (M, d) be a separable metric space. Then there exists $f : M \rightarrow c_0^+$ such that, for all $x, y \in M$, $x \neq y$, we have :*

$$d(x, y) < \|f(x) - f(y)\| < 3d(x, y).$$

This result is optimal since we observed that there is no λ -embedding from ℓ^1 into c_0^+ with $\lambda < 3$.

Corollary 6.11. *There exists $f : \ell^p \rightarrow c_0^+$ such that, for all $x, y \in M$, $x \neq y$, we have :*

$$d(x, y) < \|f(x) - f(y)\| < (1+2^p)^{1/p}d(x, y).$$

This result is optimal since we observed that there is no λ -embedding from ℓ^p into c_0^+ whenever $\lambda < (1+2^p)^{1/p}$.

Corollary 6.12. *If (M, d) is a compact space and $\varepsilon > 0$, there exists $f : M \rightarrow c_0^+$ such that, for all $x, y \in M$, $x \neq y$, we have : $d(x, y) < \|f(x) - f(y)\| < (1+\varepsilon)d(x, y)$.*

If (M, d) is a metric space such that its bounded subsets are totally bounded, there exists $f : M \rightarrow c_0^+$ such that, for all $x, y \in M$, $x \neq y$, we have : $d(x, y) < \|f(x) - f(y)\| < (2+\varepsilon)d(x, y)$.

The following result shows that we cannot replace $2 + \varepsilon$ by 2 in the above statement.

Proposition 6.13. *There exists a separable metric space M such that, for any $\lambda > 1$, M λ -embeds into c_0 but there is no 2-embedding from M into c_0^+ .*

Proof. Let (e_n) be the canonical basis of $\ell^1(\mathbb{N})$ and $F_p := \{pe_k, e_0 + pe_k; 1 \leq k \leq p\}$. We define $M = \{0, e_0\} \cup \bigcup_{p=1}^{+\infty} F_p \subset \ell^1(\mathbb{N})$. The bounded sets of M are finite, hence totally bounded, so, by Corollary 2, for any $\lambda > 1$, M λ -embeds into c_0 .

Assume now that there exists $f = (f_n) : M \rightarrow c_0^+$ such that, for all $x, y \in M$,

$$\|x - y\|_1 \leq \|f(x) - f(y)\|_\infty \leq 2\|x - y\|_1$$

Let us denote $C = \max\{\|f(0)\|_\infty, \|f(e_0)\|_\infty\}$, fix $n_0 \geq 1$ such that for all $n > n_0$, one has $f_n(0) < 1$ and $f_n(e_0) < 1$, and finally fix $p > C/2 + 1$. We claim that the mapping $\varphi : \{1, \dots, p\} \rightarrow \{0, 1\}^{n_0}$ defined by $\varphi(k) = (\mathbb{I}_{[0, C+1]}(f_n(pe_k)))_{n \leq n_0}$ is injective. This leads to a contradiction if we also have $p > 2^{n_0}$.

If $n \geq 0$ and $1 \leq k \leq p$, then

$$f_n(pe_k) \leq |f_n(pe_k) - f_n(0)| + f_n(0) \quad \text{and} \quad f_n(e_0 + pe_k) \leq |f_n(e_0 + pe_k) - f_n(e_0)| + f_n(e_0)$$

so

$$(1) \quad f_n(pe_k) \leq 2p + C \quad \text{and} \quad f_n(e_0 + pe_k) \leq 2p + C.$$

Whenever $n > n_0$, we have a better estimate:

$$f_n(pe_k) < 2p + 1 \quad \text{and} \quad f_n(e_0 + pe_k) < 2p + 1$$

So, if $n > n_0$,

$$|f_n(pe_k) - f_n(e_0 + pe_k)| \leq \max\{f_n(pe_k), f_n(e_0 + pe_k)\} < 2p + 1$$

On the other hand, if $1 \leq k \neq \ell \leq p$, we have

$$2p + 1 = \|e_0 + pe_k - pe_\ell\|_1 \leq \|f(e_0 + pe_k) - f(pe_\ell)\|_\infty,$$

hence, there exists $n \leq n_0$ such that $|f_n(e_0 + pe_k) - f_n(pe_\ell)| \geq 2p + 1$, and using the fact that $|f_n(e_0 + pe_k) - f_n(pe_k)| \leq 2$, we obtain

$$(2) \quad |f_n(pe_k) - f_n(pe_\ell)| \geq 2p - 1$$

Using (1) and (2), we obtain that either $f_n(pe_k) \leq C + 1$ and $f_n(pe_\ell) \geq 2p - 1$, or $f_n(pe_\ell) \leq C + 1$ and $f_n(pe_k) \geq 2p - 1$, hence $\mathbb{I}_{[0, C+1]}(f_n(pe_k)) \neq \mathbb{I}_{[0, C+1]}(f_n(pe_\ell))$, and φ is injective.

The proof of Theorem 6.9 is analogous to the proof of Theorem 4.1 and relies on the following two lemmas (analogous to Lemmas 4.5 and 4.6).

Lemma 6.14. *Let (M, d) be a metric space, U, V, F non empty bounded subsets of M and $\varepsilon \geq 0$. There exists $f : M \rightarrow \mathbb{R}^+$, such that $\text{Lip}(f) \leq 1$ and :*

1) *For all $x \in F$, $f(x) \leq \varepsilon$,*

2) *For all $(x, y) \in U \times V$, $f(x) - f(y) = \min \{ \delta(U, V), \max(\delta(U, F), \delta(V, F)) + \varepsilon \}$.*

Proof. Indeed, if $\delta(V, F) \leq \delta(U, F)$ and if we put $t = \min(\delta(U, V), \delta(U, F) + \varepsilon)$, the function f defined by $f(x) = \max(t - d(x, U), 0)$ satisfies Lemma 6.14.

Lemma 6.15. *Let (M, d) be a metric space with property $\pi^+(\lambda)$ with $1 < \lambda \leq 3$, $F \subset G$ be finite subsets of M and $0 < \alpha < \beta$, we set :*

$$A_+(G, \alpha) = \{ (x, y) \in M \times M; d(x, y) \geq \lambda(\max(d(x, G), d(y, G)) + \alpha) \}$$

Then there exists a finite partition $\{E_1, \dots, E_N\}$ of $A_+(G, \alpha) \setminus A_+(F, \beta)$ such that, if we denote $\pi(E_n) = U_n \times V_n$, then

$$\text{for each } n, \quad \text{diam}(E_n) < \lambda \min \{ \delta(U_n, V_n), \max(\delta(U_n, F), \delta(V_n, F)) + 2\beta \}.$$

Proof. Denote $K = \sup\{|\varphi(a)|; a \in G\}$, where φ is the control function. For all $x \in M$, we have $\varphi(x) \leq d(x, G) + K$. If $(x, y) \in A_+(G, \alpha)$, then

$$\begin{aligned} \lambda \max(d(x, G), d(y, G)) &\leq d(x, y) \leq \theta \max(\varphi(x), \varphi(y)) \\ &\leq \theta \max(d(x, G), d(y, G)) + \theta K. \end{aligned}$$

Since G is bounded and $\lambda > \theta$, we can find $B \subset M$ bounded such that $A_+(G, \alpha) \subset B \times B$. The rest of the proof follows the lines of Lemma 4.6 (show that $\delta(E_{jk}) > \max\{r_j, r_k\}$).

For the proof of Theorem 6.9, choose $\varepsilon_1 > \varphi(a_1)$, which implies, for all $x, y \in M$, $d(x, y) \leq \lambda \max(\varphi(x), \varphi(y)) < \lambda(\max(d(x, a_1), d(y, a_1)) + \varepsilon_1) := \sigma_1$.

Remark 6.16. Property $\pi(\lambda)$ characterizes the existence of a good- λ -embedding into c_0 , but we do not know if property $\pi^+(\lambda)$ characterizes the existence of a good- λ -embedding into c_0^+ . We do not know of any internal characterization of the existence of a good- λ -embedding into c_0^+ , or of the existence of a λ -embedding into c_0^+ . However, it seems very likely that if M is bounded, or if $2 < \lambda \leq 3$, then the existence of a good- λ -embedding into c_0^+ is equivalent to the fact that M has property $\pi^+(\lambda)$.

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